The Complete GCSE Guide to Right-Angled Trigonometry

Introduction:

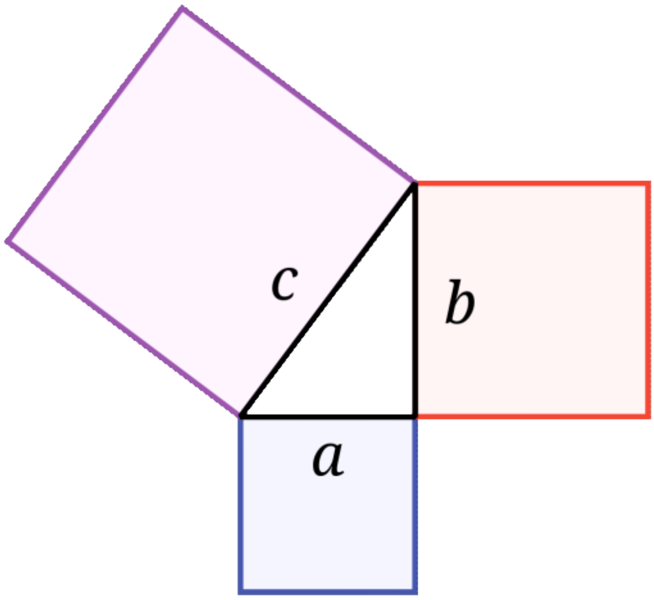
Right-angled triangles are constructed around a strict set of natural rules which began to be understood and expressed during the times of the ancient Greeks. This enables us to perform mathematical calculations to find missing sides and angles of right-angle triangles.

Pythagorean Theorem:

The Pythagorean Theorem or Pythagoras' Theorem is the first mathematical expression of the natural rules discussed above and was discovered by the philosopher Pythagoras. The theorem is as follows:

$$\boldsymbol{a^2 + b^2 = c^2}$$

This theorem describes the relationship between the three side lengths of a right-angle triangle, in which the sum of the squares of the two smaller side lengths (a and b) add to give the square of the third, largest side length (c).



We can rearrange this theorem to find the length of any of a given triangle's sides so long as we are given the length of any two of its sides. For example, to find side a, we can rearrange the equation like this: a $= \sqrt{c^2 - b^2}$.

Trigonometric Ratios:

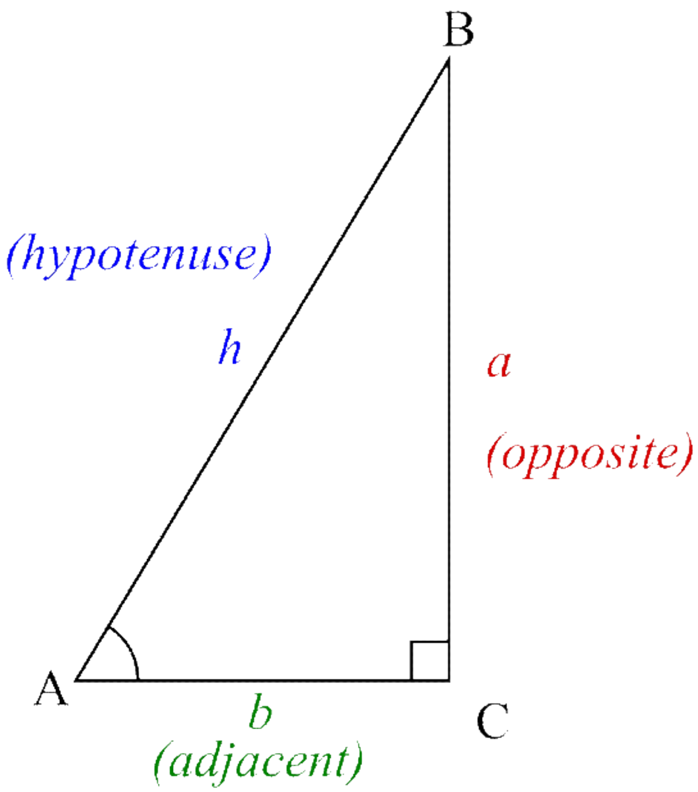
There are three basic trigonometric ratios which relate a triangle's side lengths to its angles:

$$\mathbf{sin}(\boldsymbol{\theta}) = \frac{\boldsymbol{opposite}}{\boldsymbol{hypotenuse}}$$

$$\mathbf{cos}(\boldsymbol{\theta}) = \frac{\boldsymbol{adjacent}}{\boldsymbol{hypotenuse}}$$

$$\mathbf{tan}(\boldsymbol{\theta}) = \frac{\boldsymbol{opposite}}{\boldsymbol{adjacent}}$$

In these ratios, the trigonometric functions sine ($\sin(\theta)$), cosine ($\cos(\theta)$) and tangent ($\tan(\theta)$) are used to equate the ratio of two of a triangles side lengths to an angle (θ).
The three triangle sides are the hypotenuse which ius the longest, opposite which is opposite the the angle, and adjacent which is next to the angle or neither of the other two.



In the image above, the angle (labelled A) is opposite side a and next to side b (adjacent). If the angle was to switch to position B, the opposite and adjacent sides would switch.

The trigonometric ratios can be remembered through the acronym SOH CAH TOA in which SOH stands for $\sin(\theta) = \frac{opposite}{hypotenuse}$, CAH for $\cos(\theta) = \frac{adjacent}{hypotenuse}$ and TOA for $\tan(\theta) = \frac{opposite}{adjacent}$.

The ratios can also be rearranged to find a specific angle or side. For example, to find the opposite side given the hypotenuse length and angle size we can use this equation which is a rearrangement of the SOH ratio: $opposite = \sin(\theta) \times hypotenuse$.

To find an angle, the ratios must be rearranged as follows:

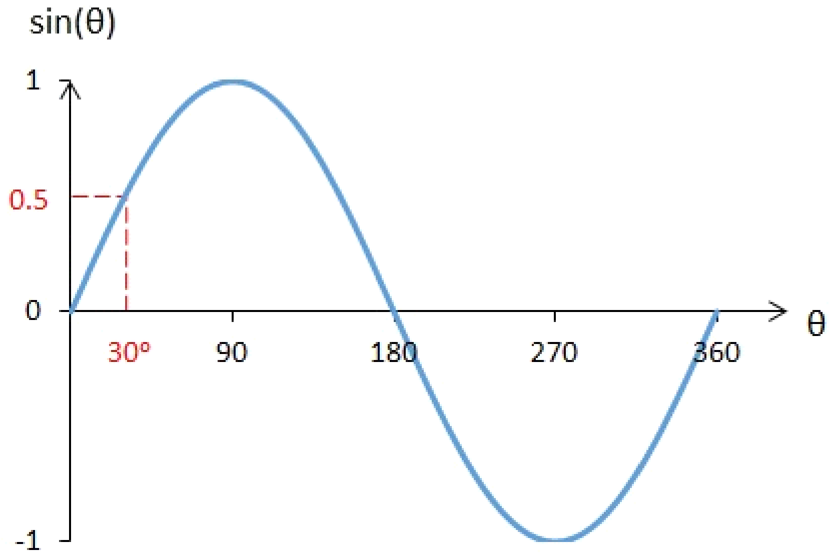
SOH	$\theta = \sin^{-1}\left(\frac{opposite}{hypotenuse}\right)$
CAH	$\theta = \cos^{-1}\left(\frac{adjacent}{hypotenuse}\right)$
TOA	$\theta = \tan^{-1}\left(\frac{opposite}{adjacent}\right)$

In the above rearrangements, the inverse trigonometric functions are used to calculate an angle from a side length ratio.

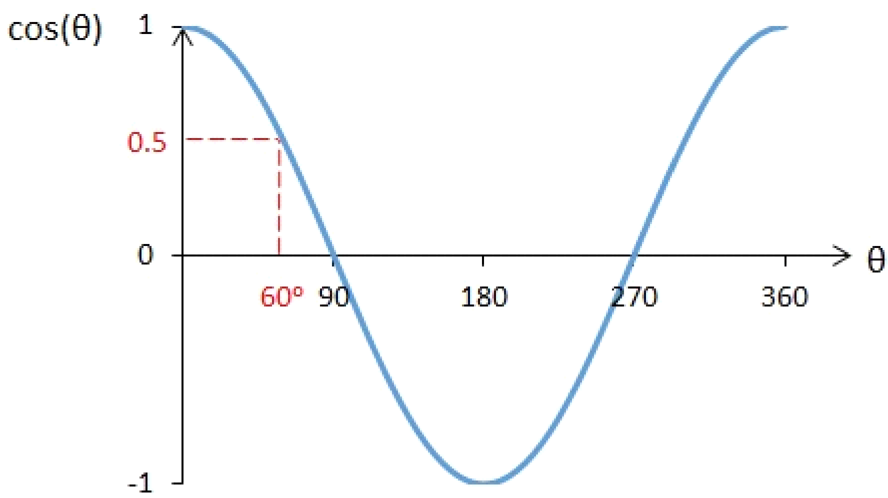
The Trigonometric Functions:

The trigonometric functions, sine, cosine and tangent, interpolate a given input (an angle or ratio from 0-1 for inverse functions) and output the value found at that position. To understand this, we first need to represent the functions graphically.

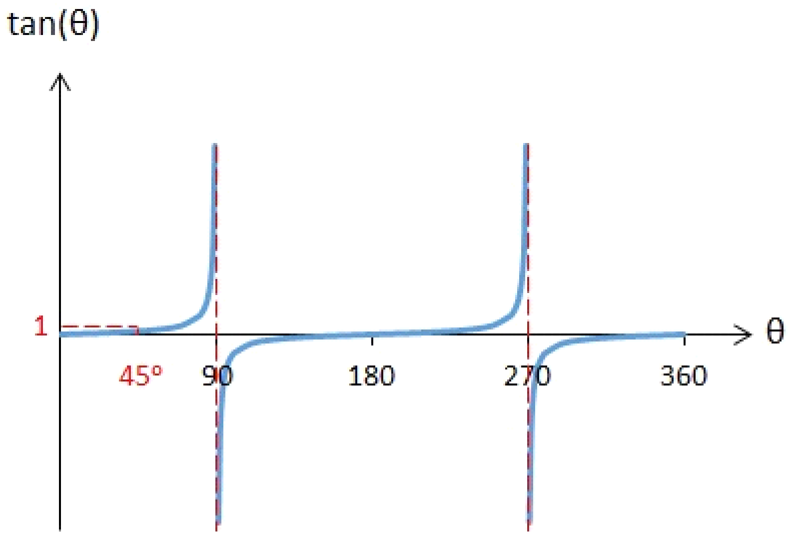
When plotted, the sine function forms a curve starting from 0 on the y-axis and reaching to 1 before turning and falling to -1, this curve continues along the x-axis from 0°-360°. This graph is called a sine wave:



The cosine function looks very similar to sine when plotted, just shifted back by 90° to begin at 1 on the y-axis. This forms a cosine wave:



Finally, the tangent function is identical to the equation $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ and so when plotted is the same as dividing a sine wave by a cosine wave, forming a tangent wave:



To calculate the output of a trigonometric function, you can find the input value on the x-axis and interpolate to find the equivalent value on the y-axis which is the output. For inverse trigonometric functions, perform the above steps in reverse, starting by finding the ratio on the y-axis and interpolating for the angle on the x-axis.

There a few easy to remember values for each function, which you are required to be able to recall for GCSE paper 1 exams, don't worry though there is an easier equivalent sequence to remember. They are as follows:

Angle	0°	30°	45°	60°	90°
$\sin(\theta)$ Value	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Easy Sequence	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$

Angle	0°	30°	45°	60°	90°
$\cos(\theta)$ Value	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Easy Sequence	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{0}}{2}$

Angle	0°	30°	45°	60°	90°
$\tan(\theta)$ Value	0	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	$\frac{2\sqrt{2}}{2\sqrt{2}} = 1$	$\frac{2\sqrt{3}}{2} = \sqrt{3}$	Undefined
Easy Sequence ($\sin(\theta) \div \cos(\theta)$)	$0 \div 1$	$\frac{1}{2} \div \frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2} \div \frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2} \div \frac{1}{2}$	$1 \div 0$